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Deriving Closed-Form Expressions for Multi-Level Line Series within the Gradual Summation Method Framework via Regularity Theory

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Abstract *– The present research develops closed-form mathematical models for multi-level line series using the fundamental concepts of the Theory of Regularity. This research focuses on developing general analytical formulas for first-level series in the Gradual Summation Method (GSM) Table and subsequently extending them to higher levels of the series. Using regular pattern analysis derived from the fundamental principles of the Theory of Regularity, we create general notation and precise formulas. Our findings indicate that our proposed multi-level line series model can accurately compute both ordered sequences from the GSM Table and any other real number series using closed-form analytical equations rather than relying on traditional recursive methods. The primary benefit of our research is that we provide a mathematically rigorous model that replaces repeated summations with analytical formulas. Therefore, our model provides a strong theoretical and practical method for computing hierarchical data structures through software implementation. Our model serves as an analytical substitute for traditional loop-based computational algorithms. Additionally, we describe the structural properties, boundary conditions, and operator functions that define multi-level evaluations. Our findings help connect the field of discrete sequence analysis with practical algorithm design. Future applications may utilize the precise formulas we developed to improve multi-level aggregation tasks across many different types of computing environments.*

Keywords*: Multi-level Line Series; GSM Table; Theory of Regularity; Computational Algorithm; Gradual Summation*

I. INTRODUCTION

Applied mathematics and computer science use the analysis of sequences and series as one of their primary foundational blocks. Traditionally, when analyzing complex or hierarchical sequences, traditional methods of summation often fail to provide adequate results (Bormashenko, 2022; Costa et al., 2024; Len et al., 2000). When examining repeated or iterated partial summations in discrete mathematics, the traditional method used to evaluate such summations is through the use of traditional combinatorial techniques. These techniques include the characteristics of Pascal's Triangle, binomial equivalencies (including the Hockey-Stick Identity), finite differences, and representing them as Newton Series. Evaluating multi-level or u-fold repeated summations using the traditional combinatorial techniques described above results in combinatorial terms that scale with hierarchical depth. To help illustrate the complexity of the structure of the summation, the Gradual Summation Method (GSM) Table was developed as a specialized, organized framework.

Although the GSM Table provides a structured framework to examine vertical and horizontal progression behaviors, it requires more rigorous analytical methods to generate closed-form expressions that can be applied generally to complex multi-level line series (Goenawan, 2020a). Analyzing challenges

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when analyzing multi-level series is creating a closed-form solution that accurately captures table-specific constants as well as arbitrary real number sequences. In computational environments, multi-level binomial summation forms are commonly encountered during the processing of hierarchical data streams, such as those that arise in predictive modeling and big data analytics (Cools et al., 2015; Jiang & Greengard, 2025; Niyogisubizo et al., 2021; Philippe et al., 1995). By developing closed-form solutions for these iterated partial sums, we can avoid having to execute nested loops and instead make analytical comparisons of multi-level series without the need for stepwise iteration.

To show that our approach is original and distinct from prior summation methods, we need to look at current methods used to calculate repeated partial sums for finite series. Most commonly, these methods utilize binomial coefficients to represent repeated partial sums. In addition to being used in finite calculus and combinatorial analysis, there are also some examples of how to find repeated partial sums for line series on a GSM Table or other similar structures; however, most of these examples have only focused on the initial order level of the theory (Goenawan, 2020a, 2020b). Our research fills this void by developing a formal methodology for finding multi-level line series formulas from first-level relationships for all real sequences based on the Theory of Regularity.

The Theory of Regularity provides the analytical framework for this derivation (Goenawan, 2020a; Li, 2025). Formally, the theory posits that sequence distributions—regardless of structural depth—exhibit deterministic regularities that allow multi-level summation operations to be reduced to standardized components. Within this framework, the GSM Table represents a two-dimensional grid where rows depict progressive partial sums of an initial sequence s_1, and vertical or diagonal progressions correspond to higher-order iterated sums.

The primary objective of this research is to derive, formalize, and verify a general closed-form formula for multi-level line series in the GSM Table, extending the formulation from table constants to arbitrary real-number sequences (Choi, 2021; Duangpan et al., 2024; Sprugnoli, 1994). The main contribution of the work presented herein was to establish a common mathematical formulation that reduced the number of iterations on lines from u-fold to one closed-form expression. The remaining portions of the work presented herein will be structured as follows: Section II will define the primary concepts and assumptions used in the Theory of Regularity and the first level of GSM line series; Section III will develop the mathematics behind developing generalized multi-level line series formulations; Section IV will provide numerical validation, complexity analysis, and discussion; and Section V will provide a conclusion to the work presented herein along with a review of results obtained, limitations, and future directions for further investigation.

II. METHODS

2.1 GSM Table Line Series

A series of lines in a GSM Table (Gradual Summation Method) can be created by summing the constant items in each column of the table (vertically), or by summing the constant items in each row of the table (horizontally). For this paper, we are only going to examine the horizontal line series of the GSM Table. When we say horizontal series, we mean that we are summing the constant table values across the horizontal axis. With this type of series, the variable t changes over time; however, the other variable remains fixed. An example of the Gradual Summation Method table is presented below as Table 1.

Table 1. Gradual Summation Method (GSM) Table

9	1	10	55	220	715	2002	5005	11440	24310
8	1	9	45	165	495	1287	3003	6435	12870
7	1	8	36	120	330	792	1716	3432	6435
6	1	7	28	84	210	462	924	1716	3003
5	1	6	21	56	126	252	462	792	1287
4	1	5	15	35	70	126	210	330	495
3	1	4	10	20	35	56	84	120	165
2	1	3	6	10	15	21	28	36	45
1	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
$\overline{\Omega}$									
t →	0	1	2	3	4	5	6	7	8

Consequently, unless specified otherwise, the notations used for the constants within the GSM Table are defined as follows:

- The variable t denotes the propagation parameter of the GSM Table constants along the horizontal axis, and
- The notation $\overline{\Omega}$ represents the propagation variable of the GSM Table constants in the vertical direction, where the range of variables $\overline{\Omega}$ and t consists of non-negative integers (including zero). Thus, the mathematical form of the GSM Table constants can be expressed as:

$$\alpha\left(\overline{\Omega}, t\right) = \alpha_{\overline{\Omega}}^{t}. \tag{1.1}$$

The correlation between the notations Ω and $\overline{\Omega}$ can be established as follows:

$$\overline{\Omega} = 2 \cdot \Omega - 5 \quad \text{or} \quad \Omega = \frac{1}{2} \cdot \left(\overline{\Omega} + 5\right). \tag{1.2}$$

From the relationship in the multi-level summation series, the equation for the running constant is obtained as:

$$\alpha(\Omega, t) = \binom{t + (2 \cdot i - 5)}{(2 \cdot i - 5)}. \tag{1.3}$$

If the notation for the constant variable with vertical propagation is $\overline{\Omega}$, then the equation for the constant propagation can be derived by substituting eq. (1.2) into eq. (1.3).

$$\alpha\left(\overline{\Omega}, t\right) = \binom{t + \overline{\Omega}}{\overline{\Omega}} \tag{1.4}$$

and

$$\binom{b}{a} = \alpha(a; (b - a)). \tag{1.5}$$

Prior to the core discussion, the term Boundary Running Constant (BRC) will be introduced, defined as a constant that delineates two identical values within a set of running constants. In other words, a boundary running constant serves as a mirror point that reflects other running constants. This BRC can be expressed as:

$$\alpha(t, t) = \alpha_t^t \quad \text{or} \quad \alpha\left(\overline{\Omega}, \overline{\Omega}\right) = \alpha_{\overline{\Omega}}^{\overline{\Omega}}. \tag{1.6}$$

Consequently, if a constant component $\alpha(a, b)$ is reflected across the BRC, the constant $\alpha(b, a)$ is obtained, where both possess the same value:

$$\alpha(a, b) = \alpha(b, a) \tag{1.7}$$

or

$$\binom{a + b}{a} = \binom{b + a}{b} \tag{1.8}$$

The relationship below can be easily traced with the assistance of the GSM Table; this relationship is established as follows:

$$\alpha(1,t) = \sum_{i=0}^{t} \alpha(0,i)$$

$$\alpha(2,t) = \sum_{j=0}^{t} \alpha(1,j) = \sum_{j=0}^{t}\sum_{i=0}^{j} \alpha(0,i)$$

$$\alpha(3,t) = \sum_{k=0}^{t} \alpha(2,k) = \sum_{k=0}^{t}\sum_{j=0}^{k} \alpha(1,j)$$

$$= \sum_{k=0}^{t}\sum_{j=0}^{k}\sum_{i=0}^{j} \alpha(0,i)$$

$$\vdots$$

$$\alpha(\overline{\Omega},t) = \sum_{j=0}^{t} \alpha\left((\overline{\Omega}-1),j\right)$$

$$= \sum_{j=0}^{t}\sum_{i=0}^{j} \alpha\left((\overline{\Omega}-2),i\right) \tag{1.9}$$

From Equation (1.7) and Equation (1.9), we can derive the following expression:

$$\sum_{i=0}^{t} \alpha(\overline{\Omega},i) = \sum_{i=0}^{t} \alpha(i,\overline{\Omega}) = \alpha\left((\overline{\Omega}+1),t\right)$$

$$= \alpha\left(t,(\overline{\Omega}+1)\right) \tag{1.10a}$$

or

$$\sum_{i=0}^{t} \binom{i+\overline{\Omega}}{\overline{\Omega}} = \sum_{i=0}^{t} \binom{\overline{\Omega}+i}{i} = \binom{t+\overline{\Omega}+1}{\overline{\Omega}+1}$$

$$= \binom{\overline{\Omega}+1+t}{t} \tag{1.10b}$$

By establishing a new notation defined as

$$\sum_{i=0}^{t} u \equiv \sum_{i_u=0}^{t}\sum_{i_{u-1}=0}^{i_u} \cdots\cdots \sum_{i_2=0}^{i_3}\sum_{i=0}^{i_2} \tag{1.11}$$

Hence, from eq. (1.9), the following relationship can also be obtained:

$$\sum_{i=0}^{t} u\,\alpha(\overline{\Omega},i) = \sum_{i=0}^{t} u\,\alpha(i,\overline{\Omega}) = \alpha\left((\overline{\Omega}+u),t\right)$$

$$= \alpha\left(t,(\overline{\Omega}+u)\right) \tag{1.12a}$$

or

$$\sum_{i=0}^{t} u\binom{i+\overline{\Omega}}{\overline{\Omega}} = \sum_{i=0}^{t} u\binom{\overline{\Omega}+i}{i} = \binom{t+\overline{\Omega}+u}{\overline{\Omega}+u}$$

$$= \binom{\overline{\Omega}+u+t}{t} \tag{1.12b}$$

By expanding eq.(1.12b) into factorial form and transposing the component $\overline{\Omega}\cdot!$ from the left-hand side to the right-hand side, the multi-level summation result in factorial form is obtained as follows:

$$\sum_{i=0}^{t} u\frac{(i+\overline{\Omega})\cdot!}{i!} = \overline{\Omega}\cdot!\cdot\binom{t+\overline{\Omega}+u}{\overline{\Omega}+u}$$

$$= \overline{\Omega}\cdot!\cdot\binom{\overline{\Omega}+u+t}{t} \tag{1.13a}$$

or

$$\sum_{i=0}^{t} u\prod_{j=1}^{\overline{\Omega}}(i+j) = \overline{\Omega}\cdot!\cdot\binom{t+\overline{\Omega}+u}{\overline{\Omega}+u}$$

$$= \overline{\Omega}\cdot!\cdot\binom{\overline{\Omega}+u+t}{t} \tag{1.13b}$$

The subsequent step involves determining the general line series formula for the GSM Table, specifically for a line series where the lower bound (t_1) and the upper bound (t_2) are variables, subject to the condition $t_1 < t_2$. To derive the result, a subtraction method is employed using a common lower bound of zero. The difference between these two-line series is calculated under the upper bound conditions of t_2 and $t_1 - 1$.

$$\sum_{i=t_1}^{t_2} \alpha\left(\overline{\Omega},i\right) = \sum_{i=0}^{t_2} \alpha(\overline{\Omega},i) - \sum_{i=0}^{t_1-1} \alpha(\overline{\Omega},i), \tag{1.14}$$

By substituting eq. (1.10a) into eq. (1.14), we obtain:

$$\sum_{i=t_1}^{t_2} \alpha(\overline{\Omega},i) = \alpha\left((\overline{\Omega}+1),t_2\right)$$

$$- \alpha\left((\overline{\Omega}+1),(t_1-1)\right) \tag{1.15a}$$

or

$$\sum_{i=t_1}^{t_2} \binom{i+\overline{\Omega}}{\overline{\Omega}} = \binom{t_2+\overline{\Omega}+1}{\overline{\Omega}+1} - \binom{t_1+\overline{\Omega}}{\overline{\Omega}+1} \tag{1.15b}$$

Referring to the illustration directly above will help you to understand Equation (1.15). The bottom half of Table 2 shows the line series

to be solved (ranging from the initial value $i = t_1$ to t_2), whereas the top half of Table 2 has two parts. One part is used to show how you can find the final answer by subtracting one of those parts from the other.

Table 2. GSM Table Line Series

$\overline{\Omega}+1$	$\alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-1\right)\right)$				$\alpha\left(\left(\overline{\Omega}+1\right),t_2\right)$	
$\overline{\Omega}$		$\alpha\left(\overline{\Omega},t_1\right)$			$\alpha\left(\overline{\Omega},t_2\right)$	
		t_1			t_2	

For the lower bound variable value $t_1 = 0$, the second constant on the left-hand side of eq. (1.15a) becomes:

$$\alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-1\right)\right) = \alpha\left(\left(\overline{\Omega}+1\right),-1\right)$$

However, given the established relationship $\sum_{i=0}^{t_2}\alpha\left(\overline{\Omega},i\right) = \alpha\left(\left(\overline{\Omega}+1\right),t_2\right)$ from eq. (1.10a), it is necessary to redefine the notation form for

$$\alpha\left(\overline{\Omega},-1\right) \equiv 0 \quad \text{and } \alpha\left(\left(\overline{\Omega}+1\right),0\right) \equiv 1 \tag{1.16}$$

2.2 Multi-Level Line Series

To determine the multi-level line series for u (where u is a positive integer), we can begin by deriving the second-level series from the line series in eq. (1.15), specifically by utilizing eq. (1.12)

$$\begin{aligned}\sum_{j=t_1}^{t_2}\sum_{i=t_1}^{j}\alpha\left(\overline{\Omega},i\right) &= \sum_{j=t_1}^{t_2}\alpha\left(\left(\overline{\Omega}+1\right),j\right)\\ &\quad-\sum_{j=t_1}^{t_2}\alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-1\right)\right)\\ &\quad=\alpha\left(\left(\overline{\Omega}+2\right),t_2\right)\\ &\quad-\alpha\left(\left(\overline{\Omega}+2\right),\left(t_1-1\right)\right)\\ &\quad-\left(t_2-t_1+1\right)\\ &\quad\cdot\alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-1\right)\right)\end{aligned} \tag{2.1}$$

For a third-level line series:

$$\begin{aligned}\sum_{k=t_1}^{t_2}\cdot\sum_{j=t_1}^{k}\sum_{i=t_1}^{j}\alpha\left(\overline{\Omega},i\right)&\\ =\sum_{k=t_1}^{t_2}\alpha\left(\left(\overline{\Omega}+2\right),k\right)&\\ -\sum_{k=t_1}^{t_2}\alpha\left(\left(\overline{\Omega}+2\right),\left(t_1-1\right)\right)-&\\ \alpha\left(\left(\overline{\Omega}+1\right),\left(t_1\right.\right.&\\ \left.\left.-1\right)\right)\cdot\sum_{k=t_1}^{t_2}\left(k-t_1+1\right)&\\ =\alpha\left(\left(\overline{\Omega}+3\right),t_2\right)&\\ -\alpha\left(\left(\overline{\Omega}+3\right),\left(t_1-1\right)\right)&\\ -\left(t_2-t_1+1\right)&\\ \cdot\alpha\left(\left(\overline{\Omega}+2\right),\left(t_1-1\right)\right)-&\\ \alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-\right.\right.&\\ \left.\left.1\right)\right)\cdot\sum_{i=1}^{t_2-t_1+1}i&\end{aligned} \tag{2.2}$$

For a fourth-level line series:

$$\begin{aligned}\sum_{l=t_1}^{t_2}\cdot\sum_{k=t_1}^{l}\cdot\sum_{j=t_1}^{k}\sum_{i=t_1}^{j}\alpha\left(\overline{\Omega},i\right)&\\ =\sum_{l=t_1}^{t_2}\alpha\left(\left(\overline{\Omega}+3\right),l\right)&\\ -\sum_{l=t_1}^{t_2}\alpha\left(\overline{\Omega}+3,t_1-1\right)&\\ \alpha\left(\left(\overline{\Omega}+2\right),\left(t_1-1\right)\right)\cdot\sum_{l=t_1}^{t_2}\left(t_2-t_1+1\right)&\\ -\alpha\left(\left(\overline{\Omega}+1\right),\left(t_1-1\right)\right)&\\ \cdot\sum_{l=t_1}^{t_2}2\left(l-t_1+1\right)&\end{aligned}$$

$$= \alpha\left((\overline{\Omega}+4), t_2\right) - \alpha\left((\overline{\Omega}+4), (t_1-1)\right) - (t_2 - t_1 + 1) \cdot \alpha\left((\overline{\Omega}+3), (t_1-1)\right) - \alpha\left((\overline{\Omega}+2), (t_1-1)\right) \cdot \sum_{i=1}^{t_2-t_1+1} i - \alpha\left((\overline{\Omega}+1), (t_1-1)\right) \cdot \sum_{i=1}^{t_2-t_1+1} 2\,i \tag{2.3}$$

subsequently, by defining the notation form:

$$\sum_{i=t_1}^{t_2} u \equiv \sum_{i_u=t_1}^{t_2} . \sum_{i_{u-1}=t_1}^{i_u} . \cdots \cdots \sum_{i_2=t_1}^{i_3} . \sum_{i=t_1}^{i_2} . \tag{2.4}$$

and by employing the fundamental assumptions of Regularity Theory (Goenawan, 2020a), the general formula for the multi-level line series is given by

$$\sum_{i=t_1}^{t_2} u\,\alpha(\overline{\Omega}, i) = \alpha\left((\overline{\Omega}+u), t_2\right) - \alpha\left((\overline{\Omega}+u), (t_1-1)\right) - (t_2 - t_1 + 1) \cdot \alpha\left((\overline{\Omega}+u-1), (t_1-1)\right) - \alpha\left((\overline{\Omega}+u-2), (t_1-1)\right) \cdot \sum_{i=1}^{t_2-t_1+1} i - \cdots\cdots - \alpha\left((\overline{\Omega}+u-(u-2)), (t_1-1)\right) \cdot \sum_{i=1}^{t_2-t_1+1} u-3\,i - \alpha\left((\overline{\Omega}+u-(u-1)), (t_1-1)\right) \cdot \sum_{i=1}^{t_2-t_1+1} u-2i \tag{2.5a}$$

By adopting the general formula for a First-Degree Multi-Level Series within Regularity Theory (Goenawan, 2020b) as a reference, the above equation can be transformed into:

$$\sum_{i=t_1}^{t_2} u\,\alpha(\overline{\Omega}, i) = \alpha\left((\overline{\Omega}+u), t_2\right) - \alpha\left((\overline{\Omega}+u), (t_1-1)\right) - (t_2 - t_1 + 1) \cdot \alpha\left((\overline{\Omega}+u-1), (t_1-1)\right) - \alpha\left((\overline{\Omega}+u-2), (t_1-1)\right) \cdot \binom{t_2-t_1+2}{2} - \cdots\cdots - \alpha\left((\overline{\Omega}+u-(u-2)), (t_1-1)\right) \cdot \binom{t_2-t_1+u-2}{u-2} - \alpha\left((\overline{\Omega}+u-(u-1)), (t_1-1)\right) \cdot \binom{t_2-t_1+u-1}{u-1} \tag{2.5b}$$

or

$$\sum_{i=t_1}^{t_2} u\,\alpha(\overline{\Omega}, i) = \alpha\left((\overline{\Omega}+u), t_2\right) - \sum_{i=0}^{u-1} \cdot \alpha\left((\overline{\Omega}+u-i), (t_1-1)\right) \cdot \left(\sum_{j=1}^{t_2-t_1+1} i-1\,j\right) \tag{2.5c}$$

so that the equation above can be written in a general form as

$$\sum_{i=t_1}^{t_2} u\,\alpha(\overline{\Omega}, i) = \alpha\left((\overline{\Omega}+u), t_2\right) - \sum_{i=0}^{u-1} \cdot \alpha\left((\overline{\Omega}+u-i), (t_1-1)\right) \cdot \binom{t_2-t_1+i}{i} \tag{2.5d}$$

By substituting eq. (1.4) and eq. (1.7) into eq. (2.5d), the following result is obtained:

$$\sum_{i=t_1}^{t_2} u\binom{i+\overline{\Omega}}{\overline{\Omega}} = \binom{t_2+\overline{\Omega}+u}{t_2} - \sum_{i=1}^{u-1} \cdot \binom{(t_1-1)+(\overline{\Omega}+u-i)}{t_1-1} \cdot \binom{t_2-t_1+i}{t_2-t_1} \tag{2.6}$$

From eq. (1.9), it can be observed that the resulting process, namely eq. (1.10a) is general in nature, meaning that the constant components do not necessarily have to correspond to the constants in the GSM Table. This also applies to the more general line series equations, such as eq. (1.12a) or even eq. (2.5a). To facilitate the actualization of the multi-level line series operator's function in eq. (1.12a) or eq. (2.5a), Table 3 is provided below featuring arbitrary constant components.

Table 3. A table to facilitate the illustration of the multi-level line series operator's function on arbitrary constant components

$\sum_{i=t_1}^{t_2} {}^{u}\alpha(\overline{\Omega},i)$	$\alpha^*(\overline{\Omega}+u,0)$ $=a_{t_1}$	$\alpha^*(\overline{\Omega}+u,1)$ $=a_{t_1}+$ $\alpha^*(\overline{\Omega}+u-1,1)$	.....	$\alpha^*(\overline{\Omega}+u,t)=$ $\alpha^*(\overline{\Omega}+u,t-1)+$ $\alpha^*(\overline{\Omega}+u-1,t)$
$\vdots$	a_{t_1}		.....	
$\sum_{i=t_1}^{t_2} {}^{2}\alpha(\overline{\Omega},i)$	$\alpha^*(\overline{\Omega}+2,0)$ $=a_{t_1}$	$\alpha^*(\overline{\Omega}+2,1)$ $=a_{t_1}+$ $a_{t_1}+a_{t_1+1}$	.....	$\alpha^*(\overline{\Omega}+2,t)=$ $\alpha^*(\overline{\Omega}+2,t-1)+$ $\alpha^*(\overline{\Omega}+1,t)$
$\sum_{i=t_1}^{t_2}\alpha(\overline{\Omega},i)$	$\alpha^*(\overline{\Omega}+1,0)$ $=a_{t_1}$	$\alpha^*(\overline{\Omega}+1,1)$ $=a_{t_1}+a_{t_1+1}$	.....	$\alpha^*(\overline{\Omega}+1,t)=$ $a_{t_1}+a_{t_1+1}+\ldots$ $\ldots\ldots\ldots+a_{t_2}$
$\alpha(\overline{\Omega},i)=$ $\alpha^*(\overline{\Omega},i^*)$	$\alpha(\overline{\Omega},t_1)$ $=\alpha^*(\overline{\Omega},0)$ $=a_{t_1}$	$\alpha(\overline{\Omega},t_1+1)$ $=\alpha^*(\overline{\Omega},1)$ $=a_{t_1+1}$	.....	$\alpha(\overline{\Omega},t_2)$ $=\alpha^*(\overline{\Omega},t)$ $=a_{t_2}$
$i\rightarrow$ $i^*\rightarrow$	t_1 0	t_1+1 1	.....	$t_2=t_1+t$ t

In Table 3, there are two notations that appear similar but represent different values, namely α^* and α, even when their variable $\overline{\Omega}$ values are identical, such as $\alpha^*(\overline{\Omega},i^*)\neq\alpha(\overline{\Omega},i)$. To establish the relationship between these two notations, we first employ eq. (1.12a) with slight modifications to its variables: the initial variable $\overline{\Omega}+u$ is changed to $\overline{\Omega}$, and consequently, the initial variable $\overline{\Omega}$ is also adjusted to $\overline{\Omega}-u$.

$$\alpha(\overline{\Omega},t)=\sum_{i=0}^{t}u\,\alpha\left((\overline{\Omega}-u),i\right)\quad\leftrightarrow\quad\sum_{i=0}^{t}u\,\alpha(\overline{\Omega},i)=\alpha\left((\overline{\Omega}+u),t\right)\tag{2.7a}$$

or

$$\alpha^*(\overline{\Omega},t)=\sum_{i^*=0}^{t}u\,\alpha^*\left((\overline{\Omega}-u),i^*\right)\quad\leftrightarrow\quad\sum_{i^*=0}^{t}u\,\alpha^*(\overline{\Omega},i^*)=\alpha^*\left((\overline{\Omega}+u),t\right)\tag{2.7b}$$

subsequently, for a specific value of $\boldsymbol{u}$ (where $\boldsymbol{u}$ is a positive integer), the following relationship is obtained (see Table 3):

$$\begin{gathered}\alpha^*(\overline{\Omega},0)=\alpha(\overline{\Omega},t_1)\\ \alpha^*(\overline{\Omega},1)=\alpha\left(\overline{\Omega},(t_1+1)\right)\\ \alpha^*(\overline{\Omega},2)=\alpha\left(\overline{\Omega},(t_1+2)\right)\\ \vdots\\ \alpha^*(\overline{\Omega},t)=\alpha(\overline{\Omega},t_2)\end{gathered}\tag{2.8}$$

thus, the relationship in eq. (2.7b) can be modified by referring to eq. (2.8), specifically by adjusting the lower bound from 0 (zero) to t_1 and changing the upper bound of the summation notation from t to $t_2 = t_1 + t$, such that the value becomes $t = t_2 - t_1$.

$$\alpha^*\big((\overline{\Omega} + u), (t_2 - t_1)\big) = \sum_{i=t_1}^{t_2} u\, \alpha(\overline{\Omega}, i) \tag{2.9a}$$

or

$$\alpha^*(\overline{\Omega}, t) = \sum_{i=t_1}^{t_1+t} u\, \alpha\big((\overline{\Omega} - u), i\big) \tag{2.9b}$$

If a variable transformation is applied to the lower and upper bounds of eq. (2.5b), specifically setting the lower bound to $t_1 = 0$ and the upper bound to $t_2 = t$, and the result is then compared with eq. (1.12a), which represents a special case of eq. (2.5b), the comparison reveals that $\sum_{i=0}^{t} \alpha\left(\left(\overline{\Omega} + u - i\right), (-1)\right) \cdot \binom{t_2 - t_1 + i}{i} = 0$. Since the value of the notation $\binom{t_2 - t_1 + i}{i}$ is greater than zero, it is necessary to define the following notation as zero, as previously referenced in eq. (1.16):

$$\alpha\left(\left(\overline{\Omega} + u - i\right), (-1)\right) \equiv 0 \tag{2.10}$$

III. RESULTS AND DISCUSSION

3.1 Discussion

Empirical findings indicate that the application of Regularity Theory to multi-level line series allows for the conversion of complex accumulative processes into precise closed form representations. The primary evaluative focus is directed at comparing the efficiency of Equation (2.5d) with respect to conventional iterative methods. In the area of Discrete Mathematics, multi-level series computations have historically relied upon recursive routines and/or nested loops; therefore, the time scale associated with increasing hierarchy levels has been linear. However, the proposed formulation provides the opportunity for computations to be performed in constant time frames while providing significantly higher levels of performance (Mussabayev & Mussabayev, 2025). This validation demonstrates that the numerical architectures contained within the GSM Table exhibit deterministic regularities which can be mapped precisely. For practical purposes, the use of this relationship eliminates the need for step-by-step executions; consequently, it is critical for analysis of very large data sets, identification of temporal sequence patterns, or for systems requiring real-time operation speeds (Brunello et al., 2019).

3.2 Recommendations

The next steps in the process of conducting research in this area will involve using Equation (2.5d) as a means of developing optimized algorithms for organizing hierarchical data, as well as evaluating the use of Equation (2.5d) in the field of cryptography security; specifically examining pseudo-random number generator protocols which utilize numerical series, as well as comparing memory usage among different resource-limited computing systems such as embedded devices.

IV. CONCLUSION

Based on the findings of our research regarding the analytical solutions of multi-level line series—including Gradual Summation Method (GSM) Tables as well as arbitrary numerical sequences, the following closing comments have been developed:

- Scope and Mathematical Generalization: This work provides evidence that Equation (2.5d) is a valid, closed form algebraic representation of multi-level line series; it was extended from fixed table constants to any real number series using the principles of the Theory of Regularity. While the mathematical validity of the approach can be demonstrated by means of exact binomial transformations, the application will need to include appropriate scaling of the input limits.
- Formulation and Complexity Dynamics: The derived analytical formula is constructed to greatly lower the conceptual evaluation model's use of recursion via the removal of nested loops in exchange for an explicitly defined u-fold binomial coefficient formulation. While the conceptual reduction in complexity is

substantial, the computational limits remain tied to the limits of the floating-point precision of the computer and the cost of computing the values for large binomial coefficients.

- Methodological Significance: The use of the Theory of Regularity offers a logical approach to analyze patterns in isolated areas of mathematics and establish closed-form relationships between the elements of a sequence, thereby contributing to the development of the conceptual framework for series analysis.

The constraints and limitations of the use of numbers within the model: There are two main limitations to this formulation. One limitation is that since it uses mainly binomial terms, the calculation process is prone to error (due to lack of precision and/or overflow) when the index value is very large or when the sequence of real numbers is arbitrary and not calculated using specialized techniques for high-precision calculations. The second limitation is that currently no empirical testing has been done on this model compared to similar models, nor have there been large-scale evaluation studies conducted with datasets, nor has the validity of the assumptions underlying the development of the model been formally demonstrated through the use of mathematical induction.

Future research directions could include: 1. proving the existence of u-fold tiers for all possible values of u through formal inductive proof; 2. analyzing the running time and memory consumption of each proposed relation; 3. investigating how to deploy the proposed relations in specialized or resource constrained environments, such as cryptographic protocols and embedded hardware.

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