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The Origin of The Basic Formula of The Fourier Series

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Abstract *– This paper discusses the Origin of the Basic Formula for the Fourier Series*

$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi t}{L}\right) + b_k \sin\left(\frac{k\pi t}{L}\right)\right]$

To obtain this model, the author derives the basic formula for the Fourier Series from Elementary Linear Algebra. In this derivation, we use the best approximation: least squares method whose details will be elaborated in this paper until we obtain the formula for the Fourier Series.

Keywords: *Fourier Series; Elementary Linear Algebra; the best approximation; least squares method*

I. INTRODUCTION

The Basic Formula for the Fourier Series was first put forward by Mr. Jean Baptiste Joseph Fourier (21 March 1768 - 16 May 1830) and he is a French mathematician and physicist who is best known for starting the investigation of the Fourier series and its application to the problem of hot currents. In deriving the Fourier Series formula, the author uses the least squares method (Best Approach) in Elementary Linear Algebra.

II. METHODS

Derivation of the Fourier Series formula using the least squares method (Best Approach) in Elementary Linear Algebra where several propositions are first introduced below.

Projection Proof: If W is a finite-dimensional linear subspace of a scalar product space V, then each $\bar{v} \in V$ can be expressed in exactly one way by $\bar{v} = \bar{w}_1 + \bar{w}_2, \bar{w}_1 \in W, \bar{w}_2$ orthogonal to W.

The Best Approach Theorem:

If (i) W is a finite-dimensional linear subspace of the scalar product space V, and (ii) $\bar{v} \in V$ and $\bar{v}_w$ are orthogonal projections of $\bar{v}$ on W, then $\forall \bar{w} \in W$ with $\bar{w} \neq \bar{v}_w$ holds $\|\bar{v} - \bar{v}_w\| < \|\bar{v} - \bar{w}\|$

Best Approach Geometry Meaning in R^3 :

Find $\bar{v} \subset W$ the point with position vector $\bar{v}_w \in W$ (linear subspace of V) whose distance from the point with position vector $\bar{v} \in V$ as small as possible is $\|\bar{v} - \bar{v}_w\|$

Meaning of Best Approach in general :

Finding $\bar{v} \subset W$ in a linear subspace with finite dimensions W ⊂ V, V scalar multiplicative space, where the distance from $\bar{v} \subset W$ is as small as possible in particular: Least squares method.

Finding a function in the linear span of the set of functions $\{\phi_0, \phi_1, \ldots, \phi_n\} \subset C[a,b]$ that is as far from $f \in C[a,b]$ as possible.

Distance is derived from scalar multiplication $< p, q >= \int_a^b p(t)\, q(t)\, dt$

III. RESULTS AND DISCUSSION

In this case, to get the Basic Formula of Fourier Series, the writer uses the least squares method (Best Approach) in Elementary Linear Algebra.

The best approximation polynomial to the power ≤ 2 using the least squares method for a function $f \in C[a,b]$ is $p(t) = a_0 + a_1 t + a_2 t^2$ with the smallest possible $\int_a^b [f(t) - p(t)]^2 dt$.

Finding the best approximation polynomial is more efficient by finding the orthogonal projection of f on P_2 with respect to scalar multiplication.

$$< p, q > = \int_a^b p(t)\, q(t)\, dt$$

First formed the orthogonal basis of P_2 with the Gram-Schmidt Process starting with $\{1, t, t^2\}$.

Order ≤ k Fourier polynomial is the best approximation trigonometry polynomial order ≤ k (PT_k) with least squares method for function f at [-π,π] or [0,2π].

$$P_k(t) = a_0 + a_1 \cos t + a_2 \cos 2t + \dots + a_k \cos kt + \dots + b_1 \sin t + \dots + b_2 \sin 2t + \dots + b_k \sin kt + \dots$$

or

$$P_k(t) = a_0 + \sum_{k=1}^{\infty} [\, a_k \cos(k\,t) + b_k \sin(k\,t)]$$

The function f defined on [-π,π] or [0,2π] will be approximated by the function $P_k(t)$ on that interval. $P_k(t)$ forms a linear subspace of C [0,2π] or C [-π,π], if the area of definition of $P_k(t)$ is confined to these intervals only.

The orthogonal basis of the linear subspace is B= $\{1, \cos t, \cos 2t, \cdots, \cos kt, \cdots, \sin t, \sin 2t, \cdots, \sin kt, \cdots\}$ because B is an orthogonal set, then according to the postulate B is linearly independent, so it is a basis for PT_k, and is an orthogonal basis for PT_k, with respect to scalarmultiplication
$< p, q > = \int_0^{2\pi} p(t)\, q(t)\, dt$ or
$< p, q > = \int_{-\pi}^{\pi} p(t)\, q(t)\, dt$.

To obtain an orthonormal basis, first determine the length of each vector in the orthogonal basis.

$$\|1\| = \sqrt{\langle p, q\rangle} = \sqrt{\int_0^{2\pi} (1)^2 dt} = \sqrt{\int_0^{2\pi} dt} = \sqrt{[t|\ _0^{2\pi}]} = \sqrt{2\pi - 0} = \sqrt{2\pi}\ ,$$

$$\|\cos k\,t\| = \sqrt{\int_0^{2\pi} (\cos(kt))^2 dt} = \sqrt{\frac{1}{2k}[\cos(kt)\cdot \sin(kt) + kt|\ _0^{2\pi}]}$$

$$= \sqrt{\frac{1}{2i}[\{\cos k\,(2\pi).\sin k\,(2\pi) + k(2\pi)\} - \{\cos 0\,.\sin 0 + k(0)\}]}$$

$$= \sqrt{\frac{1}{2k}.k(2\pi)} = \sqrt{\pi}\ ,$$

$$\|\sin(kt)\| = \sqrt{\int_0^{2\pi} (\sin(kt))^2 dt} = \sqrt{\frac{1}{2i}[-\sin(kt)\cdot \cos(kt) + kt|\ _0^{2\pi}]}$$

$$= \sqrt{\frac{1}{2i}[\{-\sin k\,(2\pi) \cos k\,(2\pi) + k(2\pi)\} - \{-\sin 0\,.\cos 0 + k(0)\}]}$$

$$= \sqrt{\frac{1}{2k}.k(2\pi)} = \sqrt{\pi}\,.$$

Thus the orthonormal basis for PT_k is =

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \dots, \frac{\cos kt}{\sqrt{\pi}}, \dots, \frac{\sin t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \dots, \frac{\sin kt}{\sqrt{\pi}}, \dots\right\}$$
$$=\{\bar{u}_0, \bar{u}_1, \bar{u}_2, \dots, \bar{u}_n, \dots, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_n, \dots\}$$

The Fourier polynomial for f(x) is the Best Approaching Trigonometric Polynomial for f(x) with the scalar multiplication above, projecting f(x) over the orthonormal basis for PT_k is

$$P(t) = \frac{\langle f, \bar{u}_0\rangle}{\langle \bar{u}_0, \bar{u}_0\rangle}\bar{u}_0 + \frac{\langle f, \bar{u}_1\rangle}{\langle \bar{u}_1, \bar{u}_1\rangle}\bar{u}_1 + \dots + \frac{\langle f, \bar{u}_k\rangle}{\langle \bar{u}_k, \bar{u}_k\rangle}\bar{u}_k + \dots + \frac{\langle f, \bar{v}_1\rangle}{\langle \bar{v}_1, \bar{v}_1\rangle}\bar{v}_1 + \dots + \frac{\langle f, \bar{v}_k\rangle}{\langle \bar{v}_n, \bar{v}_k\rangle}\bar{v}_k + \dots$$

where :

$$< \bar{u}_0, \bar{u}_0 > = \int_0^{2\pi} \bar{u}_0\, \bar{u}_0\, dt = \int_0^{2\pi} (\bar{u}_0)^2\, dt = \int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 dt = \frac{1}{2\pi}\int_0^{2\pi} dt =$$
$$= \frac{1}{2\pi}[t]_0^{2\pi} = \frac{1}{2\pi}\,(2\pi - 0) = 1\ ,$$

$$< \bar{u}_k, \bar{u}_k > = \int_0^{2\pi} \bar{u}_k\, \bar{u}_k\, dt = \int_0^{2\pi} (\bar{u}_k)^2\, dt = \int_0^{2\pi} \left(\frac{\cos\,(\text{kt})}{\sqrt{\pi}}\right)^2 dt = \int_0^{2\pi} \frac{\cos^2(kt)}{\pi}\, dt =$$
$$\frac{1}{\pi}\int_0^{2\pi} \cos^2(kt)\, dt = \frac{1}{\pi}\,(\pi) = 1\ ,$$

$$< \bar{v}_k, \bar{v}_k > = \int_0^{2\pi} \bar{v}_k\, \bar{v}_k\, dt = \int_0^{2\pi} (\bar{v}_k)^2\, dx = \int_0^{2\pi} \left(\frac{\sin\,(\text{kt})}{\sqrt{\pi}}\right)^2 dt = \int_0^{2\pi} \frac{\sin^2(\text{kt})}{\pi}\, dt =$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin^2(\text{kt})\, dt = \frac{1}{\pi}\,(\pi) = 1\ ,$$

then the Fourier Polynomial changes to

$$P(t) = \frac{\langle f, \bar{u}_0\rangle}{1}\bar{u}_0 + \frac{\langle f, \bar{u}_1\rangle}{1}\bar{u}_1 + \dots + \frac{\langle f, \bar{u}_t\rangle}{1}\bar{u}_t + \dots + \frac{\langle f, \bar{v}_1\rangle}{1}\bar{v}_1 + \dots + \frac{\langle f, \bar{v}_t\rangle}{1}\bar{v}_t + \dots$$

$$P(t) = \langle f, \bar{u}_0\rangle \bar{u}_0 + \langle f, \bar{u}_1\rangle \bar{u}_1 + \ldots + \langle f, \bar{u}_t\rangle \bar{u}_t + \ldots + \langle f, \bar{v}_1\rangle \bar{v}_1 + \ldots + \langle f, \bar{v}_t\rangle \bar{v}_t + \ldots P(t)$$

$$= \langle f, \bar{u}_0\rangle \bar{u}_0 + \sum_{i=1}^{\infty} \langle f, \bar{u}_t\rangle \bar{u}_t + \sum_{i=1}^{\infty} \langle f, \bar{v}_t\rangle \bar{v}_t$$

where:

$$< f, \bar{u}_0 > \ = \int_0^{2\pi} f(t) \left(\frac{1}{\sqrt{2\pi}}\right) dt = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t)\, dt < f, \bar{u}_0 > \bar{u}_0 = \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t)\, dt\right)\left(\frac{1}{\sqrt{2\pi}}\right) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt = \frac{\frac{1}{\pi}\int_0^{2\pi} f(t)\, dt}{2} = \frac{a_0}{2} \ ,$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\, dt$,

$\bar{u}_k = \frac{cos\ \ kt}{\sqrt{\pi}}$,

$$< f, \bar{u}_k > \ = \int_0^{2\pi} f(t) \left(\frac{cos\ (kt)}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(t)\ cos\ \ (kt) dt$$

$$\langle f, u_k\rangle u_k = \left(\frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(t)\, cos(kt)\ dt\right)\left(\frac{cos(kt)}{\sqrt{\pi}}\right) = \left(\frac{1}{\pi}\int_0^{2\pi} f(t)\, cos(kt)\ dt\right) cos(kt) = a_k\, cos(kt)$$

where $a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\ cos\ \ (kt)\, dt$,

$\bar{v}_k = \frac{sin(kt)}{\sqrt{\pi}}$,

$$< f, \bar{v}_k > = \int_0^{2\pi} f(t) \left(\frac{sin(kt)}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(t)\ sin(k\, t)\ dt \ ,$$

$$\langle f, v_k\rangle v_k = \left(\frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(t)\, sin(kt)\ dt\right)\left(\frac{sin(kt)}{\sqrt{\pi}}\right) = \left(\frac{1}{\pi}\int_0^{2\pi} f(t)\, sin(kt)\ dt\right) \sin(kt) = b_k\, sin(kt)$$

where $b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\ sin(\, kt)\, dt$.

$$P(t) = \langle f, \bar{u}_0\rangle \bar{u}_0 + \sum_{k=1}^{\infty} \langle f, \bar{u}_k\rangle \bar{u}_k + \sum_{k=1}^{\infty} \langle f, v_k\rangle \bar{v}_k$$

So

$$P(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_k\, cos(kt) + \sum_{i=1}^{\infty} b_k\, sin(kt) \ ,$$

where *k = 1,2,3,..., t*.

The Fourier polynomial is

$$P_n(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_k\, cos(kt) + \sum_{i=1}^{\infty} b_k\, sin(kt)$$

where :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\, dt$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\ cos\ \ (kt)\, dt$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\ sin(kt)\ dt$$

on interval [0,2π]

or

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, cos\ \ (kt)\, dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ sin(\, kt)\, dt$$

on interval [-π, π]

Take $k = \frac{k\pi}{\pi}$ and the denominator *π = L*, then $k = \frac{k\pi}{\pi}$ and the interval *[-L, L]* so that

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\, dt$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(t)\ cos\frac{k\pi t}{L}\ dt$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(t)\ sin\frac{k\pi t}{L}\ dt \ ,$$

on interval *[-L, L]*

Thus the Fourier Polynom turns into:

Fourier Series:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k\, cos\left(\frac{k\pi t}{L}\right) + b_k\, sin\left(\frac{k\pi t}{L}\right)\right]$$ where $k = 1,2,3,\ldots,\infty$.

IV. CONCLUSION

The derivation of the Basic Formula for the Fourier Series has been described above and the derivation is obtained :

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k\, cos\left(\frac{k\pi t}{L}\right) + b_k\, sin\left(\frac{k\pi t}{L}\right)\right]$$

In this derivation, using the least squares method (Best Approach) in Elementary Linear Algebra.

where: $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\, dt$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(t)\ cos\frac{k\pi t}{L}\ dt$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(t)\ sin\frac{k\pi t}{L}\ dt,$$

on interval *[-L, L]*

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