e-ISSN: 2686-2573
DOI: 10.21512/emacsjournal.v7i1.12726

Origin of Trigonometry Formula cos(A+B), sin(A+B), cos(A-B), sin(A-B) using the Euler Formula

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Abstract – This paper discusses the Origin of Trigonometric Formulas cos(A+B), sin(A+B), cos(A-B), and sin(A-B). To obtain the model, the writer uses Euler's formula $e^{i\theta} = cos\ \ \theta + i\ .\ sin\ \theta.$ In this derivation we will get the formulas sin(A+B), cos(A+B), sin(A-B), and cos(A-B).

Keywords: Euler Formula, trigonometric Formula; Formulas sin (A+B), cos (A+B), sin (A-B), cos (A-B)

I. INTRODUCTION

Trigonometry formulas cos(A+B), sin(A+B), cos(A-B), and sin(A-B) are generally never discussed about the origins of the formulas or formulas while in high school, but the discovery of sine cosine tangent which we often encounter in the math lesson was Mr. Ahmad ibn 'Abdallah Habash Hasib Marwazi or commonly known as Al Marwazi. He was born in Marw, Turkmenistan in 770 AD, he grew up in Baghdad and died in Samarra, Iraq in 874 AD He lived during the Abbasid caliphate al-Ma'mundan al-Mu'tasim. Al Marwazi was a Persian astronomer, geographer, and mathematician from Merv, Khorasan who first explained trigonometry ratios: Sine, Cosine, Tangent and Cotangent. During the years 825-835, Al Marwazi made observations of astronomy. In 829 he carried out research related to solar eclipses, Al Marwazi gave the first example of timing by altitude (of the sun), a method generally adopted by Muslim astronomers. (source: Swara Unsada).

II. METHODS

To find the Trigonometry Formula cos(A+B), sin(A+B), cos(A-B), and sin(A-B) in general, the origin of the formula or formula was never discussed when in high school, therefore the author is willing to convey the origin The Trigonometry Formulas cos(A+B), sin(A+B), cos(A-B) and sin(A-B). To find the cos(A+B) and sin(A+B) formulas, the first writer substitutes θ = A+B in Euler's formula and the second uses the formula: $e^{i(A+B)} = e^{i(A)}.e^{i(B)}$. Likewise, to find the cos(A-B) and sin(A-B) formulas, the first author substitutes θ = A-B in Euler's formula and the second uses the formula: $e^{i(A+B)} = e^{i(A)}.e^{i(B)}$

III. RESULTS AND DISCUSSION

In this case, to get a decrease in the Formula cos(A+B), sin(A+B), cos(A-B), and sin(A-B), the writer uses the Euler Formula Formula :

$$e^{i\theta} = \cos\ \theta + \mathrm{i}\ .\ \sin\theta$$

To find these cos(A+B) and sin(A+B) Formulas, first the writer substitutes θ=A+B in the Euler Formula :

$$e^{i\theta} = \cos\ \theta + \mathrm{i}\ .\ \sin\theta$$

$$\boldsymbol{e^{i(\mathbf{A+B})} = \cos\ (\mathbf{A+B}) + \mathbf{i}\ .\ \sin\ (\mathbf{A+B})} \qquad (1)$$

Second, $e^{i(A+B)} = e^{i(A)}.e^{i(B)}$

$$e^{i(A+B)} = [\cos(A) + \mathrm{i}\ .\ \sin(A)].[\cos(B) + \mathrm{i}\ .\ \sin(B)]$$

$e^{i(\mathrm{A+B})} =$ cos(A).cos(B) + i.cos(A).sin(B) + i.sin(A).cos(B) + i^2.sin(A).sin(B)

$e^{i(\mathrm{A+B})} =$ [cos(A).cos(B) +(-1). sin(A).sin(B)] + [i.sin(A).cos(B) + i.cos(A).sin(B)]

$\boldsymbol{e^{i(\mathbf{A+B})}}$ = **[cos (A).cos(B) - sin(A).sin(B)] + i.[sin(A).cos(B)+ cos(A).sin(B)]** (2)

If equation (1) and equation (2) are identical, then

$e^{i(\mathrm{A+B})} = e^{i(\mathrm{A+B})}$, so that

cos(A + B) = cos(*A*).cos(B) - sin(*A*).sin(B) , and

sin(A + B) = sin(A).cos(B) + cos(A).sin(B)

Example :

$$\cos 75^\circ = \cos(45^\circ + 30^\circ)$$
$$= \cos 45^\circ.\cos 30^\circ - \sin 45^\circ.\sin 30^\circ$$
$$= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{4}\sqrt{6} - \tfrac{1}{4}\sqrt{2}$$

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$
$$= \sin 45^\circ.\cos 30^\circ + \cos 45^\circ.\sin 30^\circ$$
$$= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) + \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{4}\sqrt{6} + \tfrac{1}{4}\sqrt{2}$$

$$\tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\tfrac{1}{4}\sqrt{6} + \tfrac{1}{4}\sqrt{2}}{\tfrac{1}{4}\sqrt{6} - \tfrac{1}{4}\sqrt{2}}.\frac{4}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}.\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$
$$= \frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{\left(\sqrt{6}\right)^2 - \left(\sqrt{2}\right)^2} = \frac{\left(\sqrt{6}\right)^2 + 2\left(\sqrt{6}\right)\left(\sqrt{2}\right) + \left(\sqrt{2}\right)^2}{6-2}$$
$$= \frac{6+2\sqrt{12}+2}{4} = \frac{8+4\sqrt{3}}{4} = 2+\sqrt{3}$$

To find these cos(A-B) and sin(A-B) formulas, the writer first substitutes $\theta = A - B$ in the Euler Formula: $e^{i\theta} = \cos\ \theta + \mathrm{i}\ .\ \sin\theta$

$$\boldsymbol{e^{i(A-B)} = \cos\ (A-B) + \mathbf{i}\ .\ \sin\ (A-B)} \qquad (3)$$

Second, $e^{i(A-B)} = e^{i(A)}.e^{i(-B)}$

$$e^{i(A-B)} = [\cos(A) + \mathrm{i}.\ \sin(A)].[\cos(-B) + \mathrm{i}\ .\ \sin(-B)]$$

$$e^{i(A-B)} = [\cos(A) + \mathrm{i}.\ \sin(A)].[\cos(B) - \mathrm{i}\ .\ \sin(B)]$$

$e^{i(A-B)}$ = cos(A).cos(B) - i.cos(A).sin(B) + i.sin(A).cos(B) - i^2.sin(A).sin(B)

$e^{i(A-B)}$ = [cos(A).cos(B)-(-1). sin(A).sin(B)] + [i.sin(A).cos(B) - i.cos(A).sin(B)]

$\boldsymbol{e^{i(A-B)}}$ **= [cos(A).cos(B) + sin(A).sin(B)] + i.[sin(A).cos(B) - cos(A).sin(B)]** (4)

If equation (3) and equation (4) are identical, then

$e^{i(A-B)} = e^{i(A-B)}$, so that

$\cos\ (A-B) = \cos(A).\cos(B) + \sin(A).\sin(B)$, and

$\sin(A-B) = \sin(A).\cos(B) - \cos(A).\sin(B)$

Example :

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$
$$= \cos 45^\circ . \cos 30^\circ + \sin 45^\circ . \sin 30^\circ$$
$$= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) + \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{4}\sqrt{6} + \tfrac{1}{4}\sqrt{2}$$

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$
$$= \sin 45^\circ . \cos 30^\circ - \cos 45^\circ . \sin 30^\circ$$
$$= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{4}\sqrt{6} - \tfrac{1}{4}\sqrt{2}$$

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{1}{4}\sqrt{6} - \frac{1}{4}\sqrt{2}}{\frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{2}} . \frac{4}{4}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} . \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}}$$
$$= \frac{\left(\sqrt{6} - \sqrt{2}\right)^2}{\left(\sqrt{6}\right)^2 - \left(\sqrt{2}\right)^2} = \frac{\left(\sqrt{6}\right)^2 + 2\left(\sqrt{6}\right)\left(-\sqrt{2}\right) + \left(-\sqrt{2}\right)^2}{6 - 2}$$
$$= \frac{6 - 2\sqrt{12} + 2}{4} = \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3}$$

IV. CONCLUSION

The derivation of the Trigonometry Formulas cos(A+B), sin(A+B), cos(A-B), and sin(A-B), has been described above and the derivations are obtained :

$\cos\ (A+B) = \cos(A).\cos(B) - \sin(A).\sin(B)$,

$\sin(A+B) = \sin(A).\cos(B) + \cos(A).\sin(B)$,

$\cos(A-B) = \cos(A).\cos(B) + \sin(A).\sin(B)$,

$\sin(A-B) = \sin(A).\cos(B) - \cos(A).\sin(B)$.

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